Fast Fourier Transform (Addendum)

Let $\underline{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{2N-1} \end{bmatrix}$ be a vector of length $2N$, then

$$\underline{f}_{even} = \begin{bmatrix} f_0 \\ f_2 \\ \vdots \\ f_{2N-2} \end{bmatrix} \quad \text{and} \quad \underline{f}_{odd} = \begin{bmatrix} f_1 \\ f_3 \\ \vdots \\ f_{2N-1} \end{bmatrix}$$

both have length N.

Then every Fourier coefficient of $\underline{f}_{even}$ & $\underline{f}_{odd}$ determines $\underline{\underline{4}}$ coefficients of $\underline{f}$!!

If $\mathcal{F}_k\{\underline{f}_{even}\}$ & $\mathcal{F}_k\{\underline{f}_{odd}\}$ are known, then

$$\mathcal{F}_k\{\underline{f}\} = \tfrac{1}{2}\left[\mathcal{F}_k\{\underline{f}_{even}\} + \bar{\omega}_{2N}^{k}\,\mathcal{F}_k\{\underline{f}_{odd}\}\right]$$

$$\mathcal{F}_{N+k}\{f\} = \tfrac{1}{2}\left[\mathcal{F}_k\{\underline{f}_{even}\} - \bar{\omega}_{2N}^{k}\,\mathcal{F}_k\{f_{odd}\}\right]$$

$$\mathcal{F}_{N-k}\{f\} = \mathcal{F}_{2N-(N+k)}\{f\} = \overline{\mathcal{F}_{N+k}\{f\}}$$

$$\mathcal{F}_{2N-k}\{f\} = \overline{\mathcal{F}_k\{f\}}$$

conjugate

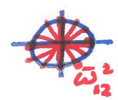
EX: Suppose $\underline{f}$ has length 12 and

$$\mathcal{F}_2\{\underline{f}_{even}\} = 2+2i$$

$$\mathcal{F}_2\{\underline{f}_{odd}\} = 4-2i$$

Compute coefficients of $\underline{f}$.

First we will rotate $\mathcal{F}_2\{\underline{f}_{odd}\}$.



$$\left.\begin{array}{l}\text{length } 2N=12 \\ \text{coefficient } k=2\end{array}\right\}\ \text{rotation } \bar{\omega}_{12}^{2} = \tfrac{1}{2}(1-\sqrt{3}\,i)$$

$$\bar{\omega}_{12}^{2}\,\mathcal{F}_2\{\underline{f}_{odd}\} = \tfrac{1}{2}(1-\sqrt{3}i)\cdot(4-2i) = (2-\sqrt{3}) - (1+2\sqrt{3})i$$

Now compute coefficients.

$$\mathcal{F}_2\{\underline{f}\} = \tfrac{1}{2}\left[(2+2i) + ((2-\sqrt{3}) - (1+2\sqrt{3})i)\right] = \tfrac{1}{2}(4-\sqrt{3}) + \tfrac{1}{2}(1-2\sqrt{3})i$$

$$\mathcal{F}_8\{f\} = \mathcal{F}_{6+2}\{f\} = \tfrac{1}{2}\left[(2+2i) - ((2-\sqrt{3}) - (1+2\sqrt{3})i)\right] = \tfrac{1}{2}(\sqrt{3}) + \tfrac{1}{2}(3+2\sqrt{3})i$$

conjugate

$$\mathcal{F}_4\{f\} = \mathcal{F}_{6-2}\{f\} = \tfrac{1}{2}(\sqrt{3}) - \tfrac{1}{2}(3+2\sqrt{3})i$$

$$\mathcal{F}_{10}\{f\} = \mathcal{F}_{12-2}\{f\} = \tfrac{1}{2}(4-\sqrt{3}) - \tfrac{1}{2}(1-2\sqrt{3})i$$